Mixed Hodge structure of affine hypersurfaces [1]

Hossein Movasati

Abstract

In this article we give an algorithm which produces a basis of the n-th de Rham cohomology of the affine smooth hypersurface $f^{-1}(t)$ compatible with the mixed Hodge structure, where f is a polynomial in $n+1$ variables and satisfies a certain regularity condition at infinity (and hence has isolated singularities). As an application we show that the notion of a Hodge cycle in regular fibers of f is given in terms of the vanishing of integrals of certain polynomial n-forms in $\mathbb{C}^{n+1}$ over topological n-cycles on the fibers of f. Since the n-th homology of a regular fiber is generated by vanishing cycles, this leads us to study Abelian integrals over them. Our result generalizes and uses the arguments of J. Steenbrink in [St77] for quasi-homogeneous polynomials.

0   Introduction

To study the monodromy and some numerical invariants of a local holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at 0, E. Brieskorn in [Br70] introduced a $\mathcal{O}_{\mathbb{C},0}$ module H' and the notion of Gauss-Manin connection on H'. Later J. Steenbrink [St76], inspired by P. Deligne's theory of mixed Hodge structures (see [De71] and two others with the same title) on algebraic varieties defined over complex numbers and W. Schmid's limit Hodge structure (see [Sc73]) associated to a fibration with projective manifolds, introduced the notion of the mixed Hodge structure for a germ of a singularity f. A different construction of such a mixed Hodge structure was also given by A. Varchenko in [Va80] using the asymptotic of integrals of holomorphic forms over vanishing cycles.

In the case of a polynomial f in $\mathbb{C}^{n+1}$, on the one hand the n-th cohomology of a regular fiber carries Deligne's mixed Hodge structure and on the other hand we have the Brieskorn module H' of f which contains the information of the n-th de Rham cohomology of regular fibers. The variation of mixed Hodge structures in such situations is studied by Steenbrink and Zucker (see [SZ85]). In this article we define two filtrations on H' based on the mixed Hodge structure of the regular fibers of f. At the beginning my purpose was to find explicit descriptions of arithmetic properties of Hodge cycles for hypersurfaces in projective spaces. Such descriptions for CM-Abelian varieties are well-known but in the case of hypersurfaces we have only descriptions for Fermat varieties (see [Sh79]). As an application we will see that it is possible to write down the property of being a Hodge cycle in terms of the vanishing of certain integrals over cycles generated by vanishing cycles. Such integrals also appear in the context of holomorphic foliations/differential equations (see [Mo04-1, Mo04-2] and the references there). The first advantage of this approach is that we can write some consequences of the Hodge conjecture in terms of periods (see the example at the end of this Introduction and §7). We explain below the results in this article.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \in \mathbb{N}^{n+1}$ and assume that the greatest common divisor of all the α_i's is one. We consider $\mathbb{C}[x] := \mathbb{C}[x_1, x_2, \ldots, x_{n+1}]$ as a graded algebra with $deg(x_i) = \alpha_i$. A polynomial $f \in \mathbb{C}[x]$ is called a quasi-homogeneous polynomial of degree d with respect to the grading α if f is a linear combination of monomials of the type

---

$x^\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}}, \alpha.\beta := \sum_{i=1}^{n+1} \alpha_i \beta_i = d$. For an arbitrary polynomial $f \in \mathbb{C}[x]$ one can write in a unique way $f = \sum_{i=0}^{d} f_i$, $f_d \neq 0$, where f_i is a quasi-homogeneous polynomial of degree i. The number d is called the degree of f. Set $\mathrm{S}ing(f) := \{\frac{\partial f}{\partial x_i} = 0,\ i = 1, 2, \ldots, n+1\}$.

Let us be given a polynomial $f \in \mathbb{C}[x]$. We assume that f is a (weighted) strongly tame polynomial. In this article this means that there exist natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathbb{N}$ such that $\mathrm{S}ing(g) = \{0\}$, where $g = f_d$ is the last quasi-homogeneous piece of f in the graded algebra $\mathbb{C}[x]$, $\mathrm{d}eg(x_i) = \alpha_i$. Looking at f as a rational function in the weighted projective space (see §1) we will see that the strongly tameness condition on f implies that the polynomial f has isolated singularities, i.e. $\mathrm{S}ing(f)$ is a discrete set in $\mathbb{C}^{n+1}$.

We choose a basis $x^I = \{x^\beta \mid \beta \in I\}$ of monomials for the Milnor $\mathbb{C}$-vector space $V_g := \mathbb{C}[x]/ < \frac{\partial g}{\partial x_i} \mid i = 1, 2, \ldots, n+1 >$ and define

$$w_i := \frac{\alpha_i}{d},\ 1 \le i \le n+1,\ \eta := (\sum_{i=1}^{n+1} (-1)^{i-1} w_i x_i \widehat{dx_i}), A_\beta := \sum_{i=1}^{n+1} (\beta_i + 1) w_i,\ \eta_\beta := x^\beta \eta,\ \beta \in I$$

(1)

where $\widehat{dx_i} = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{n+1}$. It turns out that x^I is also a basis of V_f and so f and g have the same Milnor numbers (see §6). We denote it by μ. We denote by $C = \{c_1, c_2, \ldots, c_r\} \subset \mathbb{C}$ the set of critical values of f and $L_c := f^{-1}(c),\ c \in \mathbb{C}$. The strongly tameness condition on f implies that the fibers $L_c, c \in \mathbb{C} \backslash C$ are connected and the function f is a C^∞ fiber bundle on $\mathbb{C} \backslash C$ (see §1). Let $\Omega^i, i = 1, 2, \ldots, n+1$ be the set of polynomial differential i-forms in $\mathbb{C}^{n+1}$. The Brieskorn module

$$H' = H'_f := \frac{\Omega^n}{df \wedge \Omega^{n-1} + d\Omega^{n-1}}$$

of f is a $\mathbb{C}[t]$-module in a natural way: $t.[\omega] = [f\omega]$, $[\omega] \in H'$. If there is no danger of confusion we will not write the brackets. A direct generalization of the topological argument in [Ga98] in the case $n = 1$ implies that H' is freely generated by the forms $\eta_\beta,\ \beta \in I$ (see Proposition 1 for an algebraic proof).

Using vanishing theorems and the Atiyah-Hodge-Grothendieck theorem on the de Rham cohomology of affine varieties (see [Gr66]), we see that H' restricted to each regular fiber $L_c,\ c \in \mathbb{C} \backslash C$ is isomorphic to the n-th de Rham cohomology of L_c with complex coefficients. The Gauss-Manin connection associated to the fibration f on H' turns out to be a map

$$\nabla : H' \to H'_C$$

satisfying the Leibniz rule, where for a set $\tilde{C} \subset \mathbb{C}$ by $H'_{\tilde{C}}$ we mean the localization of H' on the multiplicative subgroup of $\mathbb{C}[t]$ generated by $t - c,\ c \in \tilde{C}$ (see §3). Using the Leibniz rule one can extend ∇ to a function from $H'_{\tilde{C}}$ to itself. Here $\tilde{C}$ is any subset of $\mathbb{C}$ containing C.

The mixed Hodge structure $(W_\bullet, F^\bullet)$ of $H^n(L_c, \mathbb{C})$ is defined by Deligne in [De71] using the hypercohomology interpretation of the cohomology of L_c and the sheaf of meromorphic forms with logarithmic poles. It is natural to define a double filtration $(W_\bullet, F^\bullet)$ in H' as follows: $W_m H'$, $m \in \mathbb{Z}$ (resp. $F^k H'$, $k \in \mathbb{Z}$) consists of elements $\psi \in H'$ such that the restriction of ψ on all L_c's except a finite number of them belongs to $W_m H^n(L_c, \mathbb{C})$ (resp. $F^k H^n(L_c, \mathbb{C})$). In connection to the work of Steenbrink and Zucker, we mention that on $\mathbb{C} \backslash C$ we have the variation of mixed Hodge structures $H^n(L_t, \mathbb{C})$, $t \in \mathbb{C} \backslash C$. The Brieskorn module H' for a strongly tame polynomial gives a free extension to $\mathbb{C}$ of the underlying

free $\mathcal{O}_{\mathbb{C}\backslash C}$-module. Here we identify coherent sheaves on $\mathbb{C}$ with finite modules over $\mathbb{C}[t]$ by taking the global sections. Therefore, the mentioned filtrations of H' in this text are given by the maximal extensions as $\mathbb{C}[t]$-submodules of the Brieskorn module. Since in our situation H' is freely generated of finite rank, they are also freely generated sub-modules of H'. Their rank is equal to to the dimensions of the mixed Hodge structure of a regular fiber of f. Note that we do not know yet whether $Gr_F^k Gr_m^W H'$, $k \in \mathbb{Z}$, $m = n, n+1$ are freely generated $\mathbb{C}[t]$-modules. In the same way we define $(W_\bullet, F^\bullet)$ of the localization of H' over multiplicative subgroups of $\mathbb{C}[t]$. In this article we prove that:

Theorem 1. *Let $b \in \mathbb{C}\backslash C$ be a regular value of $f \in \mathbb{C}[x]$. If f is a (weighted) strongly tame polynomial then $Gr_m^W H' = 0$ for $m \neq n, n+1$ and there exist a map $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ and $C \subset \tilde{C} \subset \mathbb{C}$ such that $b \notin \tilde{C}$ and*

(2)
$$\nabla^k \eta_\beta, \ \beta \in I, \ A_\beta = k$$

form a basis of $Gr_F^{n+1-k} Gr_{n+1}^W H'_{\tilde{C}}$ and the forms

(3)
$$\nabla^k \eta_\beta, A_\beta + \frac{1}{d} \leq k \leq A_\beta + \frac{d_\beta}{d}$$

form a basis of $Gr_F^{n+1-k} Gr_n^W H'_{\tilde{C}}$, where $\nabla^k = \nabla \circ \nabla \circ \cdots \circ \nabla$ k times.

The numbers d_β and the set $\tilde{C}$ are calculated from a monomial basis of the Jacobian of the homogenization of $f - b$ (see Lemma 5) and hence they are not unique and may depend on the choice of b. For a generic b one can put $d_\beta = d - 1$ but this is not the case for all b's (see Example 2). In [Mo06-2] the equality $\tilde{C} = C$ is shown for many examples of f in two variables and for a suitable choice of d_β's. For those examples a similar theorem is proved as above for the Brieskorn module rather than its localization. This has many applications in the theory of Abelian integrals in differential equations (see [Ga02, Mo04-2]). When $f = g$ is a quasi-homogeneous polynomial of degree d with an isolated singularity at $0 \in \mathbb{C}^{n+1}$ our result can be obtained from J. Steenbrink's result in [St77] using the residue theory adapted to Brieskorn modules (see Lemma 2, §4). In this case $C = \{0\}$ and any two regular fibers are biholomorphic. We have $d_\beta = d - 1$, $\forall \beta \in I$, $\nabla(\eta_\beta) = \frac{A_\beta}{t} \eta_\beta$ and so $\nabla^k \eta_\beta = \frac{A_\beta (A_\beta - 1) \cdots (A_\beta - k + 1)}{t^k} \eta_\beta$. In this case we get the following stronger statement: $\eta_\beta, A_\beta = k \in \mathbb{N}$ form a basis of $Gr_F^{n+1-k} Gr_{n+1}^W H'$ and $\eta_\beta, A_\beta \notin \mathbb{N}, -[-A_\beta] = k$ form a basis of $Gr_F^{n+1-k} Gr_n^W H'$.

One may look at the fibration $f = t$ as an affine variety X defined over the function field $\mathbb{C}(t)$ and interpret Theorem 1 as the existence of mixed Hodge structure on the de Rham cohomology of X (see [Gr66] and also [Pi97]). However, we note that the Brieskorn module is something finer than the de Rham cohomology of X; for instance if we do not have the tameness property H' may not be finitely generated.

One of the initial motivations for me to get theorems like Theorem 1 was in obtaining the property of being a Hodge cycle in terms of the vanishing of explicit integrals of polynomial n-forms in $\mathbb{C}^{n+1}$. In the case n even, a cycle in $H_n(L_c, \mathbb{Z})$, $c \notin C$ is called a Hodge cycle if its image in $H_n(\widehat{L_c}, \mathbb{Z})$ is a Hodge cycle, where $\widehat{L_c}$ is the smooth compactification of L_c. Since the mixed Hodge structure on $H^n(L_c, \mathbb{C})$ is independent of the compactification and the map $i : H^n(\widehat{L_c}, \mathbb{C}) \to H^n(L_c, \mathbb{C})$ induced by the inclusion $L_c \subset \widehat{L_c}$ is a weight zero morphism of mixed Hodge structures, this definition does not depend on the compactification of L_c. Moreover, $Gr_n^W H^n(L_c, \mathbb{C})$ in the case $\alpha_1 = \cdots = \alpha_{n+1} = 1$ coincides with

the primitive cohomology of the canonical compactification and hence, we capture all the Hodge cycles contained in the primitive cohomology (via Poincaré duality).

Corollary 1. *In the situation of Theorem 1, a cycle $\delta_c \in H_n(L_c, \mathbb{Z}), c \in \mathbb{C}\backslash\tilde{C}$ is Hodge if and only if*

$$(\frac{\partial^k}{\partial t^k} \int_{\delta_t} \eta_\beta) \mid_{t=c} = 0, \ \forall (\beta, k) \in I_h,$$

where $I_h = \{(\beta, k) \in I \times \mathbb{Z} \mid A_\beta + \frac{1}{d} \leq k \leq A_\beta + \frac{d_\beta}{d}, \ A_\beta \notin \mathbb{N}, \ k \leq \frac{n}{2}\}$ and $\{\delta_t\}_{t\in(\mathbb{C},c)}$ is a continuous family of cycles in the fibers of f which is obtained by using a local topological trivialization of f around c.

Note that

$$(\frac{\partial^k}{\partial t^k} \int_{\delta_t} \eta_\beta) \mid_{t=c} = \int_{\delta_c} \nabla^k \eta_\beta = \sum_{\beta \in I} p_{\beta,k}(c) \int_{\delta_c} \eta_\beta$$

where $\nabla^k \eta_\beta = \sum_{\beta \in I} p_{\beta,k} \eta_\beta, \ p_{\beta,k} \in \mathbb{C}[t]_C$, and the forms $\nabla^k \eta_\beta, (\beta, k) \in I_h$ form a basis of $F^{\frac{n}{2}+1} H'_{\tilde{C}} \cap W_n H'_{\tilde{C}}$. Also $H_n(L_c, \mathbb{Z})$ is generated by a distinguished set of vanishing cycles (see [DN01, AGV88]) and one may be interested in constructing such a distinguished set of vanishing cycles, try to carry out explicit integration and hence obtain more explicit descriptions of Hodge cycles. For an $\omega \in H'$ the function $h(t) = \int_{\delta_t} \omega$ extends to a multivalued function on $\mathbb{C}\backslash C$ and satisfies a Picard-Fuchs equation with possible poles at C. For a quasi-homogeneous polynomial $f = g$ the Picard-Fuchs equation associated to η_β is $t\frac{\partial h}{\partial t} - A_\beta.h = 0$. For the example $f = x_1^3 + x_2^3 + \cdots + x_5^3 - x_1 - x_2$ which has a non-trivial variation of Hodge structures using SINGULAR (see §7) we get the following fact: For all $c \in \mathbb{C} - \{\pm\frac{4}{3\sqrt{3}}, 0\}$ a cycle $\delta \in H_4(L_c, \mathbb{Z})$ is Hodge if and only if

$$(972c^2 - 192) \int_\delta x_1 x_2 \eta + (-405c^3 - 48c) \int_\delta x_2 \eta + (-405c^3 - 48c) \int_\delta x_1 \eta + (243c^4 - 36c^2 + 64) \int_\delta \eta = 0$$

Since the Hodge conjecture is proved for cubic hypersurfaces of dimension 4 by C. Clemens, J. P. Murre and S. Zucker (see [Zu77]), we conclude that all the integrals appearing in the above equality divided by π^2 are algebraic numbers (see §7). In [MR06] we have used this idea to compute the values of the Gauss hypergeometric series at certain algebraic points.

Let us explain the structure of the article. In §1 we recall some terminology on weighted projective spaces. In §2 we explain the idea that to be able to give descriptions of Hodge cycles in terms of integrals one must consider them with support in affine varieties and then use Theorem 1 and get the property of being a Hodge cycle in terms of the vanishing of integrals. In §3 we introduce two Brieskorn modules H' and H'' associated to a polynomial f and the notion of Gauss-Manin connection on them. The version of Gauss-Manin connection we use here comes from the context of differential equations (see [Mo04-2]) and the main point about it is that we can iterate it. In §4 we see how the iteration of an element ω of H'' by the Gauss-Manin connection is related to the residue of $\omega/(f - c)^k, k \in \mathbb{N}$ in the regular fiber L_c of f. §5 is dedicated to a generalization of a theorem of Griffiths (see [Gr69]) to weighted projective spaces by J. Steenbrink (see [St77]). The main point in this section is Theorem 2. What is new is an explicit basis of the underlying cohomology. In §6 we prove Theorem 1. §7 is dedicated to some examples.

When the first draft of this paper was written M. Schulze told me about his article [Sc05] in which he gives an algorithm to calculate a good $\mathbb{C}[\nabla^{-1}]$-basis of the Brieskorn module for strongly tame polynomials. For the moment the only thing which I can say is that in the case $f = g$ the set $\{\eta_\beta, \beta \in I\}$ is also a good basis of the $\mathbb{C}[\nabla^{-1}]$-module

H' because $t\eta_\beta = (A_\beta + 1)\nabla^{-1}\eta_\beta$, $\forall \beta \in I$. In particular, any strongly tame polynomial in the sense of this article has the same monodromy at infinity as g and so the spectrum of H' is equal to $\{A_\beta + 1, \beta \in I\}$. The generalization of the result of this article for a tame polynomial in the sense of [Sa99] or a Lefschetz pencil (see [Mo06-1]) would be a nice challenge. Note that the pair $(W_\bullet, F^\bullet)$ defined on H' is different form the mixed Hodge structure constructed in [Sa99] (using also the Gauss-Manin connection), in which the weight filtration is due to the monodromy at infinity.

One can compute ∇, d_β's and calculate every element of H' as a $\mathbb{C}[t]$-linear combination of η_β's. These are done in the library `foliation.lib` written in SINGULAR and is explained in [Mo06-2]. This paper is devoted to the applications of the existence of such a basis in differential equations. Also for many examples it is shown that by modification of Theorem 1 one can get a basis of H' compatible with $W_\bullet H'$ and $F^\bullet H'$.

1 Weighted projective spaces

In this section we recall some terminology on weighted projective spaces. For more information on weighted projective spaces the reader is referred to [Do82, St77].

Let n be a natural number and $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ be a vector of natural numbers whose greatest common divisor is one. The multiplicative group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ in the following way:

$$(X_1, X_2, \ldots, X_{n+1}) \to (\lambda^{\alpha_1} X_1, \lambda^{\alpha_2} X_2, \ldots, \lambda^{\alpha_{n+1}} X_{n+1}), \ \lambda \in \mathbb{C}^*$$

We also denote the above map by λ. The quotient space

$$\mathbb{P}^\alpha := \mathbb{C}^{n+1}/\mathbb{C}^*$$

is called the projective space of weight α. If $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 1$ then $\mathbb{P}^\alpha$ is the usual projective space $\mathbb{P}^n$ (Since n is a natural number, $\mathbb{P}^n$ will not mean a zero dimensional weighted projective space). One can give another interpretation of $\mathbb{P}^\alpha$ as follow: Let $G_{\alpha_i} := \{e^{\frac{2\pi\sqrt{-1}m}{\alpha_i}} \mid m \in \mathbb{Z}\}$. The group $\Pi_{i=1}^{n+1} G_{\alpha_i}$ acts discretely on the usual projective space $\mathbb{P}^n$ as follows:

$$(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}), [X_1 : X_2 : \cdots : X_{n+1}] \to [\epsilon_1 X_1 : \epsilon_2 X_2 : \cdots : \epsilon_{n+1} X_{n+1}]$$

The quotient space $\mathbb{P}^n / \Pi_{i=1}^{n+1} G_{\alpha_i}$ is canonically isomorphic to $\mathbb{P}^\alpha$. This canonical isomorphism is given by

$$[X_1 : X_2 : \cdots : X_{n+1}] \in \mathbb{P}^n / \Pi_{i=1}^{n+1} G_{\alpha_i} \to [X_1^{\alpha_1} : X_2^{\alpha_2} : \cdots : X_{n+1}^{\alpha_{n+1}}] \in \mathbb{P}^\alpha$$

Let d be a natural number. The polynomial (resp. the polynomial form) ω in $\mathbb{C}^{n+1}$ is weighted homogeneous of degree d if

$$\lambda^*(\omega) = \lambda^d \omega, \ \lambda \in \mathbb{C}^*$$

For a polynomial g this means that

$$g(\lambda^{\alpha_1} X_1, \lambda^{\alpha_2} X_2, \ldots, \lambda^{\alpha_{n+1}} X_{n+1}) = \lambda^d g(X_1, X_2, \ldots, X_{n+1}), \ \forall \lambda \in \mathbb{C}^*$$

Let g be an irreducible polynomial of (weighted) degree d. The set $g = 0$ induces a hypersurface D in $\mathbb{P}^\alpha$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$. If g has an isolated singularity at $0 \in \mathbb{C}^{n+1}$ then Steenbrink has proved that D is a V-manifold/quasi-smooth variety. A V-manifold may be singular but it has many common features with smooth varieties (see [St77, Do82]).

For a polynomial form ω of degree dk, $k \in \mathbb{N}$ in $\mathbb{C}^{n+1}$ we have $\lambda^* \frac{\omega}{g^k} = \frac{\omega}{g^k}$ for all $\lambda \in \mathbb{C}^*$. Therefore, $\frac{\omega}{g^k}$ induce a meromorphic form on $\mathbb{P}^\alpha$ with poles of order k along D. If there is no confusion we denote it again by $\frac{\omega}{g^k}$. The polynomial form

$$(4) \qquad \eta_\alpha = \sum_{i=1}^{n+1} (-1)^{i-1} \alpha_i X_i \widehat{dX_i}$$

where $\widehat{dX_i} = dX_1 \wedge \cdots \wedge dX_{i-1} \wedge dX_{i+1} \wedge \cdots \wedge dX_{n+1}$, is of degree $\sum_{i=1}^{n+1} \alpha_i$.

Let $\mathbb{P}^{(1,\alpha)} = \{[X_0 : X_1 : \cdots : X_{n+1}] \mid (X_0, X_1, \cdots, X_{n+1}) \in \mathbb{C}^{n+2}\}$ be the projective space of weight $(1, \alpha)$, $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$. One can consider $\mathbb{P}^{(1,\alpha)}$ as a compactification of $\mathbb{C}^{n+1} = \{(x_1, x_2, \ldots, x_{n+1})\}$ by putting

$$(5) \qquad x_i = \frac{X_i}{X_0^{\alpha_i}}, \ i = 1, 2, \cdots, n+1$$

The projective space at infinity $\mathbb{P}_\infty^\alpha = \mathbb{P}^{(1,\alpha)} - \mathbb{C}^{n+1}$ is of weight $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$.

Let f be the strongly tame polynomial of (weighted) degree d in the introduction and g be its last quasi-homogeneous part. Now we can look at f as a rational function on $\mathbb{P}^{(1,\alpha)}$ and the fibration $f = t$ as a pencil in $\mathbb{P}^{(1,\alpha)}$ with the axis $\{g = 0\} \subset \mathbb{P}_\infty^\alpha$. Note that $\mathbb{P}_\infty^\alpha$ itself is a fiber of this pencil. This implies that the closure $\overline{L}_c$ of $L_c := f^{-1}(c)$ in $\mathbb{P}^{(1,\alpha)}$ intersects $\mathbb{P}_\infty^\alpha$ transversally in the sense of V-manifolds. In particular, 1. f has connected fibers because f at the infinity has connected fibers 2. f has only isolated singularities because every algebraic variety of dimension greater than zero in $\mathbb{P}^{(1,\alpha)}$ intersects $\mathbb{P}_\infty^\alpha$. After making a blow-up along the axis $\{g = 0\} \subset \mathbb{P}_\infty^\alpha$ and using Ehresmann's fibration theorem one concludes that f is C^∞ fiber bundle over $\mathbb{C} \backslash C$.

2 Hodge cycles

Let M be a smooth projective complex manifold of dimension n. The cohomologies of M with complex coefficients carry the so called Hodge decomposition

$$(6) \qquad H^m(M, \mathbb{C}) = H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{1,m-1} \oplus H^{0,m}$$

Using de Rham cohomology

$$H^m(M, \mathbb{C}) \cong H^m_{deR}(M) := \frac{Z_d^m}{dA^{m-1}}$$

we have $H^{p,q} \cong \frac{Z_d^{p,q}}{dA^{p+q-1} \cap Z_d^{p,q}}$, where A^m (resp. Z_d^m, $Z_d^{p,q}$) is the set of C^∞ differential m-forms (resp. closed m-forms, closed (p,q)-forms) on M (with this notation one has the

canonical inclusions $H^{p,q} \to H^m(M, \mathbb{C})$ and one can prove (6) using harmonic forms, see M. Green's lectures [GMV94], p. 14). The Hodge filtration is defined

$$G^p := F^p H^m(M, \mathbb{C}) = H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{p,m-p}$$

Let m be an even natural number and $Z = \sum_{i=1}^s r_i Z_i$, where Z_i, $i = 1, 2, \ldots, s$ is a subvariety of M of complex dimension $\frac{m}{2}$ and $r_i \in \mathbb{Z}$. Using a resolution map $\tilde{Z}_i \to M$, where $\tilde{Z}_i$ is a complex manifold, one can define an element $\sum_{i=1}^s r_i[Z_i] \in H_m(M, \mathbb{Z})$ which is called an algebraic cycle (see [BH61]). Since the restriction to Z of a (p, q)-form with $p + q = m$ and $p \neq \frac{m}{2}$ is identically zero, an algebraic cycle δ has the following property:

$$\int_\delta G^{\frac{m}{2}+1} = 0$$

A cycle $\delta \in H_m(M, \mathbb{Z})$ with the above property is called a Hodge cycle. The assertion of the Hodge conjecture is that if we consider the rational homologies then a Hodge cycle $\delta \in H_n(M, \mathbb{Q})$ is an algebraic cycle, i.e. there exist subvarieties $Z_i \subset M$ of dimension $\frac{m}{2}$ and rational numbers r_i such that $\delta = \sum r_i[Z_i]$. The difficulty of this conjecture lies in constructing varieties just with their homological information.

Now let U be a quasi-projective smooth variety, $U \subset M$ its compactification in the projective variety M such that $N := M - U$ is a divisor with normal crossings (see [De71] 3.2) and

$$i : H_m(U, \mathbb{Z}) \to H_m(M, \mathbb{Z})$$

be the map induced by the inclusion $U \subset M$. For instance, $L_c = f^{-1}(c), c \in \mathbb{C} \backslash C$ of the previous section with $m = n$ is an example of such a quasi-projective smooth variety whose compactification divisor has only one irreducible component. We are interested to identify Hodge cycles in the image of i.

Remark 1. Let M be a hypersurface of even dimension n in the projective space $\mathbb{P}^{n+1}$. By the first Lefschetz theorem $H_m(M, \mathbb{Z}) \cong H_m(\mathbb{P}^{n+1}, \mathbb{Z})$, $m < n$ and so the only interesting Hodge cycles are in $H_n(M, \mathbb{Z})$. For a general hypeplane section N of M, the long exact sequence of the pair (M, U), where U is the complement of N in M, gives rise to the isomorphism $H^n_{\text{prim}}(M, \mathbb{Q}) \cong W_n H^n(U, \mathbb{Q})$ induced by the inclusion $U \subset M$. This implies that we capture all Hodge cycles in the primitive cohomology (using Poincaré duality).

The mixed Hodge structure of $H^m(U, \mathbb{Q})$ consists of two filtrations

(7) $$0 = F^{m+1} \subset F^m \subset \cdots \subset F^1 \subset F^0 = H^m(U, \mathbb{C})$$

$$0 = W_{m-1} \subset W_m \subset W_{m+1} \subset \cdots W_{m+a-1} \subset W_{m+a} = H^m(U, \mathbb{C}), 1 \leq a \leq m$$

where W is defined over $\mathbb{Q}$, i.e. it is defined on $H^n(U, \mathbb{Q})$ and we have complexified it. The number a is the minimum of m and the number of irreducible components of N. Therefore, it is 1 for L_c. The first is the Hodge filtration and the second is the weight filtration. The Hodge filtration induces a filtration on $Gr_a^W := W_a/W_{a-1}$ and we set

(8) $$Gr_F^b Gr_a^W := F^b Gr_a^W / F^{b+1} Gr_a^W = \frac{(F^b \cap W_a) + W_{a-1}}{(F^{b+1} \cap W_a) + W_{a-1}}, a, b \in \mathbb{Z}$$

Let $r : H^m(M, \mathbb{C}) \to H^m(U, \mathbb{C})$ be induced by inclusion. We have $W_m = r(H^m(M, \mathbb{C}))$ and r is a weight zero morphism of mixed Hodge structures. It is strict and in particular

$$r(G^{\frac{m}{2}+1}) = F^{\frac{m}{2}+1} \cap Im(r) = F^{\frac{m}{2}+1} \cap W_m$$

7

(see for instance [KK98] for definitions). Now let us be given a cycle $\delta \in H_m(U, \mathbb{Z})$ whose image in $H_m(M, \mathbb{Z})$ is Hodge. The condition of being a Hodge cycle translate into a property of δ using the mixed Hodge structure of $H^m(U, \mathbb{C})$ as follows

$$\int_{i(\delta)} G^{\frac{m}{2}+1} = \int_\delta r(G^{\frac{m}{2}+1}) =$$

(9)
$$\int_\delta F^{\frac{m}{2}+1} \cap W_m = 0$$

Definition 1. *A cycle $\delta \in H_m(U, \mathbb{Z})$ is called Hodge if (9) holds, where $F^{\frac{m}{2}+1}$ (resp. W_m) is the $(\frac{m}{2}+1)$-th (resp. m-th) piece of the Hodge filtration (resp. weight filtration) of the mixed Hodge structure of $H^m(U, \mathbb{Q})$.*

All the elements in the kernel of i are Hodge cycles and we call them trivial Hodge cycles.

3 Global Brieskorn modules

In this section we introduce two Brieskorn modules H' and H'' associated to a polynomial f and the notion of Gauss-Manin connection on them. In the usual definition of Gauss-Manin connection for n-th cohomology of the fibers of f, if we take global sections and then compose it with the vector field $\frac{\partial}{\partial t}$ in $\mathbb{C}$ then we obtain our version of Gauss-Manin connection.

Let f be the strongly tame polynomial in the introduction. Multiplying by f defines a linear operator on

(10)
$$V_f := \frac{\mathbb{C}[x]}{< \frac{\partial f}{\partial x_i} \mid i = 1, 2, \ldots, n+1 >}$$

which is denoted by A. In the previous section we have seen that f has isolated singularities and so V_f is a $\mathbb{C}$-vector space of finite dimension μ, where μ is the sum of local Milnor numbers of f, and eigenvalues of A are exactly the critical values of f (see for instance [Mo04-2], Lemma 1.1). Let $S(t) \in \mathbb{C}[t]$ be the minimal polynomial of A, i.e. the polynomial with the minimum degree and with the leading coefficient 1 such that $S(A) \equiv 0$ as a function from V_f to V_f

$$S(f) = \sum_{i=1}^{n+1} p_i \frac{\partial f}{\partial x_i}, \ p_i \in \mathbb{C}[x]$$

or equivalently

(11)
$$S(f)dx = df \wedge \eta_f, \ \eta_f = \sum_{i=1}^{n+1} (-1)^{i-1} p_i \widehat{dx_i}$$

From now on we fix an η_f with the above property. To calculate $S(t)$ we may start with the characteristic polynomial $S(t) = det(A - tI)$, where I is the $\mu \times \mu$ identity matrix and we have fixed a monomial basis of V_f and have written A as a matrix. This S has the property (11) but it is in general useless from computational point of view (see §7). Note that if f has rational coefficients then S has also rational coefficients. The polynomial S has only zeros at critical values C of f.

The global Brieskorn modules are

$$H'' = \frac{\Omega^{n+1}}{df \wedge d\Omega^{n-1}}, \ H' = \frac{df \wedge \Omega^n}{df \wedge d\Omega^{n-1}}$$

They are $\mathbb{C}[t]$-modules. Multiplication by t corresponds to the usual multiplication of differential forms with f. The Gauss-Manin connection

$$\nabla : H' \to H'', \ \nabla([df \wedge \omega]) = [d\omega]$$

is a well-defined function and satisfies the Leibniz rule

(12) $$\nabla(p\omega) = p\nabla(\omega) + p'\omega, \ p \in \mathbb{C}[t], \ \omega \in H'$$

where p' is the derivation with respect to t. Let H'_C (resp. H''_C and $\mathbb{C}[t]_C$) be the localization of H' (resp. H'' and $\mathbb{C}[t]$) on the multiplicative subgroup of $\mathbb{C}[t]$ generated by $\{t - c, c \in C\}$. An element of H'_C is a fraction ω/p, $\omega \in H'$, $p \in \mathbb{C}[t]$, $\{p = 0\} \subset C$. Two such fractions ω/p and $\tilde{\omega}/\tilde{p}$ are equal if $\tilde{p}\omega = p\tilde{\omega}$. We have $\frac{H''}{H'} = V_f$ and so $S.H'' \subset H'$. This means that the inclusion $H' \subset H''$ induces an equality $H'_C = H''_C$. We denote by H_C the both side of the equality. Let $\tilde{\Omega}^i$ denote the set of rational differential i- forms in $\mathbb{C}^{n+1}$ with poles along the L_c, $c \in C$. The canonical map $H_C \to \frac{\tilde{\Omega}^{n+1}}{df \wedge d\tilde{\Omega}^n}$ is an isomorphism of $\mathbb{C}[t]_C$-modules and this gives another interpretation of H_C. One extends ∇ as a function from H_C to itself by

(13) $$\nabla(\frac{[df \wedge \omega]}{p}) = [d(\frac{\omega}{p(f)})] = \frac{p[d\omega] - p'[df \wedge \omega]}{p^2}, \ p \in \mathbb{C}[t], \ [df \wedge \omega] \in H'$$

This is a natural extension of ∇ because it satisfies

$$\nabla(\frac{\omega}{p}) = \frac{p\nabla\omega - p'\omega}{p^2}, \ p \in \mathbb{C}[t], \ \omega \in H_C$$

Lemma 1. *We have*

$$\nabla([Pdx]) = \frac{[(Q_P - P.S'(f))dx]}{S}, \ P \in \mathbb{C}[x]$$

where

$$Q_P = \sum_{i=1}^{n+1}(\frac{\partial P}{\partial x_i}p_i + P\frac{\partial p_i}{\partial x_i})$$

Proof.

$$\begin{aligned}
\nabla([Pdx]) &= \nabla(\frac{[df \wedge P\eta_f]}{S}) = [d(\frac{P\eta_f}{S(f)})] = \frac{[S(f)d(P\eta_f) - S'(f)Pdf \wedge \eta_f]}{S^2} \\
&= \frac{[d(P\eta_f) - S'(f)Pdx]}{S} = \frac{[dP \wedge \eta_f + P.d\eta_f - P.S'(f)dx]}{S} \\
&= \frac{[(Q_P - P.S'(f))dx]}{S}
\end{aligned}$$

$\square$

It is better to have in mind that the polynomial Q_P is defined by the relation $d(P\eta_f) = Q_P dx$. In the next section we will use the iterations of Gauss-Manin connection, $\nabla^k = \nabla \circ \nabla \circ \cdots \circ \nabla, k$ times. To be able to calculate them we need the following operators

$$\nabla_k : H'' \to H'', \ k = 0, 1, 2, \ldots$$

$$\nabla_k(\omega) = \nabla(\frac{\omega}{S(t)^k})S(t)^{k+1} = S(t)\nabla(\omega) - k.S'(t)\omega$$

For $\omega = Pdx$ we obtain the formula

$$\nabla_k(Pdx) = (Q_P - (k+1)S'(t)P)dx$$

We show by induction on k that

(14) $$\nabla^k = \frac{\nabla_{k-1} \circ \nabla_{k-2} \circ \cdots \circ \nabla_0}{S(t)^k}$$

The case $k = 1$ is trivial. If the equality is true for k then

$$\begin{aligned}
\nabla^{k+1} &= \nabla \circ \nabla^k = \nabla(\frac{\nabla_{k-1} \circ \nabla_{k-2} \circ \cdots \circ \nabla_0}{S(t)^k}) \\
&= \frac{\nabla_0 \circ \nabla_{k-1} \circ \nabla_{k-2} \circ \cdots \circ \nabla_0 - kS'(t)\nabla_{k-1} \circ \nabla_{k-2} \circ \cdots \circ \nabla_0}{S(t)^{k+1}} \\
&= \frac{\nabla_k \circ \nabla_{k-1} \circ \cdots \circ \nabla_0}{S(t)^{k+1}}
\end{aligned}$$

The Brieskorn module $H' = \frac{\Omega^n}{df \wedge \Omega^{n-1} + d\Omega^{n-1}}$ defined in the introduction is isomorphic to the one in this section by the map $[\omega] \to [df \wedge \omega]$. The inverse of the canonical isomorphism $H'_C \to H''_C$ is denoted by $\omega \in H''_C \to \frac{\omega}{df} \in H'_C$. The Gauss-Manin connection with this notation can be written in the form

$$\nabla : H' \to H'_C, \ \nabla(\omega) = \frac{d\omega}{df} := \frac{\omega_1}{S(t)}$$

where $S(f)d\omega = df \wedge \omega_1$. In the literature one also calls $\frac{d\omega}{df}$ the Gelfand-Leray form of $d\omega$. Looking in this way it turns out that

(15) $$df \wedge \nabla\omega = \nabla(df \wedge \omega), \ \forall \omega \in H'$$

Let U be an small open set in $\mathbb{C}\backslash C$, $\delta_t \in H_n(L_t, \mathbb{Z})$, $t \in U$ be a continuous family of cycles and $\omega \in H'$. The main property of the Gauss-Manin connection is

(16) $$\frac{\partial}{\partial t} \int_{\delta_t} \omega = \int_{\delta_t} \nabla\omega$$

Recall the notations introduced for a quasi-homogeneous polynomial $f = g$ in Introduction. For this f $S(t) = t$ and η_f is η in (1). This means that $fdx = df \wedge \eta$. Since this equality is linear in f it is enough to check it for monomials $x^\alpha, \alpha.w = 1$.

$$dx^\alpha \wedge \eta = (\sum_{i=1}^{n+1} \alpha_i \frac{x^\alpha}{x_i}dx_i) \wedge (\sum_{i=1}^{n+1}(-1)^{i-1}w_i x_i \widehat{dx_i}) = (\alpha.w)x^\alpha dx = x^\alpha dx$$

10

We have also $d\eta = (w.1)dx$.

$$d\eta_\beta = dx^\beta \wedge \eta + x^\beta d\eta = (\beta.w)x^\beta dx + (w.1)x^\beta dx = A_\beta x^\beta dx = \frac{A_\beta}{f}df \wedge (x^\beta \eta)$$

which implies that $\nabla\eta_\beta = \frac{A_\beta}{t}\eta_\beta$ (In the same way on can check that $\nabla(x^\beta dx) = \frac{(A_\beta - 1)}{t}x^\beta dx$). This implies that $\frac{\partial}{\partial t}\int_{\delta_t}\eta_\beta = \frac{A_\beta}{t}\int_{\delta_t}\eta_\beta$. Therefore there exists a constant number C depending only on η_β and δ_t such that $\int_{\delta_t}\eta_\beta = Ct^{A_\beta}$. One can take a branch of t^{A_β} so that $C = \int_{\delta_1}\eta_\beta$.

4 Residue map on the Brieskorn module

Let us be given a closed submanifold N of pure real codimension c in a manifold M. The Leray (or Thom-Gysin) isomorphism is

$$\tau : H_{k-c}(N, \mathbb{Z})\tilde{\rightarrow}H_k(M, M - N, \mathbb{Z})$$

holding for any k, with the convention that $H_s(N) = 0$ for $s < 0$. Roughly speaking, given $x \in H_{k-c}(N)$, its image by this isomorphism is obtained by thickening a cycle representing x, each point of it growing into a closed c-disk transverse to N in M (see for instance [Ch96] p. 537). Let N be a connected codimension one submanifold of the complex manifold M of dimension n. Writing the long exact sequence of the pair $(M, M - N)$ and using τ we obtain:

(17) $\cdots \rightarrow H_{n+1}(M, \mathbb{Z}) \rightarrow H_{n-1}(N, \mathbb{Z}) \xrightarrow{\sigma} H_n(M - N, \mathbb{Z}) \xrightarrow{i} H_n(M, \mathbb{Z}) \rightarrow \cdots$

where σ is the composition of the boundary operator with τ and i is induced by inclusion. Let $\omega \in H^n(M - N, \mathbb{C}) := H_n(M - N, \mathbb{Z})^* \otimes \mathbb{C}$, where $H_n(M - N, \mathbb{Z})^*$ is the dual of $H_n(M - N, \mathbb{Z})$. The composition $\omega \circ \sigma : H_{n-1}(N, \mathbb{Z}) \rightarrow \mathbb{C}$ defines a linear map and its complexification is an element in $H^{n-1}(N, \mathbb{C})$. It is denoted by $\mathrm{Res}_N(\omega)$ and called the residue of ω in N. We consider the case in which ω in the n-th de Rham cohomology of $M - N$ is represented by a meromorphic C^∞ differential form ω' in M with poles of order at most one along N. Let $f_\alpha = 0$ be the defining equation of N in a neighborhood U_α of a point $p \in N$ in M and write $\omega' = \omega_\alpha \wedge \frac{df}{f}$. For two such neighborhoods U_α and U_β with non empty intersection we have $\omega_\alpha = \omega_\beta$ restricted to N. Therefore we get a $(n-1)$-form on N which in the de Rham cohomology of N represents $\mathrm{Res}_N\omega$ (see [Gr69] for details). This is called Poincaré residue. The residue map Res_N is a morphism of weight -2 of mixed Hodge structures, i.e.

$$\mathrm{Res}_N(W_p H^n(M - N, \mathbb{C})) \subset \mathrm{Res}_N(W_{p-2}H^{n-1}(N, \mathbb{C})), \; p \in \mathbb{Z}$$

$$\mathrm{Res}_N(F^q H^n(M - N, \mathbb{C})) \subset \mathrm{Res}_N(F^{q-1}H^{n-1}(N, \mathbb{C})), \; q \in \mathbb{Z}$$

We fix a regular value $c \in \mathbb{C}\backslash C$. To each $\omega \in H''_C$ we can associate the residue of $\frac{\omega}{(t-c)^k}$ in L_c which is going to be an element of $H^n(L_c, \mathbb{C})$. This map is well-defined because

$$\frac{df \wedge d\omega}{p(f)(f - c)^k} = d(\frac{df \wedge \omega}{p(f)(f - c)^k}), \; p \in \mathbb{C}[t], \; \omega \in \Omega^{n-1}$$

11

Lemma 2. *For $\omega \in H''$ and $k = 2, 3, \ldots$ the forms $\frac{\omega}{(t-c)^k}$ and $\frac{\nabla\omega}{(k-1)(t-c)^{k-1}}$ have the same residue in L_c. In particular the residue of $\frac{\omega}{(t-c)^k}$ in L_c is the restriction of $\frac{\nabla^{k-1}\omega}{(k-1)!df}$ to L_c and the residue of $\frac{df \wedge \omega}{(t-c)^k}$, $\omega \in H'$ in L_c is the restriction of $\frac{\nabla^{k-1}\omega}{(k-1)!}$ to L_c.*

Proof. This Lemma is well-known in the theory of Gauss-Manin systems (see for instance [ST85]). We give an alternative proof in the context of this article. We have

$$\nabla\left(\frac{\omega}{(t-c)^{k-1}}\right) = \frac{\nabla(\omega)}{(t-c)^{k-1}} - (k-1)\frac{\omega}{(t-c)^k}$$

According to (13), the left hand side corresponds to an exact form and so it does not produce a residue in L_c. This proves that first part. To obtain the second part we repeat $k-1$ times the result of the first part on $\frac{\omega}{(t-c)^k}$ and we obtain $\frac{\nabla^{k-1}\omega}{(k-1)!(t-c)}$. Now we take the Poincaré residue and obtain the second statement. The third statement is a consequence of the second and the identity (15). □

Note that the residue of $\frac{\omega}{t-c}$, $\omega \in H''$ in L_c coincides with the restriction of $\frac{\omega}{df} \in H'_C$ to L_c.

5 Griffiths-Steenbrink Theorem

This section is dedicated to a classic theorem of Griffiths in [Gr69]. Its generalization for quasi-homogeneous spaces is due to Steenbrink in [St77]. In both cases there is not given an explicit basis of the Hodge structure of the complement of a smooth hypersurface. This is the main reason to put Theorem 2 in this article. Recall the notations of §1.

Lemma 3. *For a monomial x^β with $A_\beta = k \in \mathbb{N}$, the meromorphic form $\frac{x^\beta dx}{(f-t)^k}$ has a pole of order one at infinity and its Poincaré residue at infinity is $\frac{X^\beta \eta_\alpha}{g^k}$.*

Proof. Let us write the above form in the homogeneous coordinates (5). We use $d(\frac{X_i}{X_0^{\alpha_i}}) = X_0^{-\alpha_i}dX_i - \alpha_i X_i X_0^{-\alpha_i-1}dX_0$ and

$$
\begin{aligned}
\frac{x^\beta dx}{(f-t)^k} &= \frac{(\frac{X_1}{X_0^{\alpha_1}})^{\beta_1}\cdots(\frac{X_{n+1}}{X_0^{\alpha_{n+1}}})^{\beta_{n+1}}d(\frac{X_1}{X_0^{\alpha_1}})\wedge\cdots\wedge d(\frac{X_{n+1}}{X_0^{\alpha_{n+1}}})}{(f(\frac{X_1}{X_0^{\alpha_1}},\cdots,\frac{X_{n+1}}{X_0^{\alpha_{n+1}}})-t)^k}\\
&= \frac{X^\beta \eta_{(1,\alpha)}}{X_0^{(\sum_{i=1}^{n+1}\beta_i\alpha_i)+(\sum_{i=1}^{n+1}\alpha_i)+1-kd}(X_0\tilde{F}-g(X_1,X_2,\cdots,X_{n+1}))^k}\\
&= \frac{X^\beta \eta_{(1,\alpha)}}{X_0(X_0\tilde{F}-g(X_1,X_2,\cdots,X_{n+1}))^k}\\
&= \frac{dX_0}{X_0}\wedge\frac{X^\beta\eta_\alpha}{(X_0\tilde{F}-g)^k}
\end{aligned}
$$

The last equality is up to forms without pole at $X_0 = 0$. The restriction of $\frac{X^\beta\eta_\alpha}{(X_0\tilde{F}-g)^k}$ to $X_0 = 0$ gives us the desired form. □

Theorem 2. *(Griffiths-Steenbrink) Let $g(X_1, X_2, \cdots, X_{n+1})$ be a weighted homogeneous polynomial of degree d, weight $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ and with an isolated singularity at $0 \in \mathbb{C}^{n+1}$(and so $D = \{g = 0\}$ is a V-manifold). We have*

$$H^n(\mathbb{P}^\alpha - D, \mathbb{C}) \cong \frac{H^0(\mathbb{P}^\alpha, \Omega^n(*D))}{dH^0(\mathbb{P}^\alpha, \Omega^{n-1}(*D))}$$

and under the above isomorphism

(18) $$Gr_F^{n+1-k} Gr_{n+1}^W H^n(\mathbb{P}^\alpha - D, \mathbb{C}) := F^{n-k+1}/F^{n-k+2} \cong$$

$$\frac{H^0(\mathbb{P}^\alpha, \Omega^n(kD))}{dH^0(\mathbb{P}^\alpha, \Omega^{n-1}((k-1)D)) + H^0(\mathbb{P}^\alpha, \Omega^n((k-1)D))}$$

where $0 = F^{n+1} \subset F^n \subset \cdots \subset F^1 \subset F^0 = H^n(\mathbb{P}^\alpha - D, \mathbb{C})$ is the Hodge filtration of $H^n(\mathbb{P}^\alpha - D, \mathbb{C})$. Let $\{X^\beta \mid \beta \in I\}$ be a basis of monomials for the Milnor vector space

$$\mathbb{C}[X_1, X_2, \cdots, X_{n+1}]/ < \frac{\partial g}{\partial X_i} \mid i = 1, 2, \ldots, n+1 >$$

A basis of (18) is given by

(19) $$\frac{X^\beta \eta_\alpha}{g^k}, \ \beta \in I, \ A_\beta = k$$

where η_α is given by (4).

Recall that if D is normal crossing divisor in a projective variety M then $H^m(M - D, \mathbb{C}) \cong \mathbb{H}^m(M, \Omega^\bullet(\log D)), m \geq 1$ and the i-th piece of the Hodge filtration of $H^m(M - D, \mathbb{C})$ under this isomorphism is given by $\mathbb{H}^m(M, \Omega^{\bullet \geq i}(\log D))$ (see [De71]). By definition we have $F^0/F^1 \cong H^n(M, \mathcal{O}_M)$ and so in the situation of the above theorem $F^0 = F^1$. Note that in the above theorem the residue map $r : H^n(\mathbb{P}^\alpha - D, \mathbb{C}) \to H^{n-1}(D, \mathbb{C})_0$ is an isomorphism of Hodge structures of weight -2, i.e. it maps the k-th piece of the Hodge filtration of $H^n(\mathbb{P}^\alpha - D, \mathbb{C})$ to the $(k-1)$-th piece of the Hodge filtration of $H^{n-1}(D, \mathbb{C})_0$. Here the sub index 0 means the primitive cohomology.

Proof. The first part of this theorem for usual projective spaces is due to Griffiths [Gr69]. The generalization for quasi-homogeneous spaces is due to Steenbrink [St77]. The essential ingredient in the proof is Bott's vanishing theorem for quasi-homogeneous spaces: Let $L \in H^1(\mathbb{P}^\alpha, \mathcal{O}^*)$ be a line bundle on $\mathbb{P}^\alpha$ with $c(\pi^*L) = k$, where $\mathbb{P}^n \to \mathbb{P}^\alpha$ is the canonical map. Then $H^p(\mathbb{P}^\alpha, \Omega^q_{\mathbb{P}^\alpha} \otimes L) = 0$ except possibly in the case $p = q$ and $k = 0$, or $p = 0$ and $k > q$, or $p = n$ and $k < q - b$.

The proof of the second part which gives an explicit basis of Hodge filtration is as follows: We consider $\mathbb{P}^\alpha$ as the projective space at infinity in $\mathbb{P}^{(1,\alpha)}$. According to Lemma 3 for $f = g$ the residue of the form $\frac{x^\beta dx}{(g-1)^k}$ at $g = 1$ is (19). Now we use Lemma 5 of [St77]. This lemma says that the residue of $\frac{x^\beta dx}{(g-1)^k}$ at infinity form a basis of (18). $\square$

6 An explicit basis of $H_{\tilde{C}}$

In this section we prove Theorem 1. In the following by homogeneous we mean weighted homogeneous with respect to fixed weights $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$.

Let f be the polynomial in the introduction, $f = f_0 + f_1 + f_2 + \cdots + f_{d-1} + f_d$ be its homogeneous decomposition in the graded ring $\mathbb{C}[x_1, x_2, \ldots, x_{n+1}]$, $\deg(x_i) = \alpha_i$ and $g := f_d$ the last homogeneous part of f. Let also $F = f_0 x_0^d + f_1 x_0^{d-1} + \cdots + f_{d-1} x_0 + g$ be the homogenization of f.

Lemma 4. *The set x^I generates freely the $\mathbb{C}[x_0]$-module*

$$V := \mathbb{C}[x_0, x] / < \frac{\partial F}{\partial x_i} \mid i = 1, 2, \ldots, n+1 >$$

Proof. First we prove that x^I generates V as a $\mathbb{C}[x_0]$-module. We write the expansion of $P(x_0, x) = \sum_{i=0}^{k} x_0^i P_i(x) \in \mathbb{C}[x_0, x]$ in x_0 and so it is enough to prove that every element $P \in \mathbb{C}[x]$ can be written in the form

$$(20) \qquad P = \sum_{\beta \in I} C_\beta x^\beta + \sum_{i=1}^{n+1} Q_i \frac{\partial F}{\partial x_i}, \quad C_\beta \in \mathbb{C}[x_0], \ Q_i \in \mathbb{C}[x_0, x]$$

Since x^I is a basis of V_g, we can write

$$(21) \qquad P = \sum_{\beta \in I} c_\beta x^\beta + \sum_{i=1}^{n+1} q_i \frac{\partial g}{\partial x_i}, \quad c_\beta \in \mathbb{C}, \ q_i \in \mathbb{C}[x]$$

We can choose q_i's so that

$$(22) \qquad \deg(q_i) + \deg(\frac{\partial g}{\partial x_i}) \leq \deg(P)$$

If this is not the case then we write the non-trivial homogeneous equation of highest degree obtained from (21). Note that $\frac{\partial g}{\partial x_i}$ is homogeneous. If some terms of P occur in this new equation then we have already (22). If not we subtract this new equation from (21). We repeat this until getting the first case and so the desired inequality. Now we have $\frac{\partial g}{\partial x_i} = \frac{\partial F}{\partial x_i} - x_0 \sum_{j=0}^{d-1} \frac{\partial f_j}{\partial x_i} x_0^{d-j-1}$ and so

$$(23) \qquad P = \sum_{\beta \in I} c_\beta x^\beta + \sum_{i=1}^{n+1} q_i \frac{\partial F}{\partial x_i} - x_0 \left(\sum_{i=1}^{n+1} \sum_{j=0}^{d-1} q_i \frac{\partial f_j}{\partial x_i} x_0^{d-j-1} \right)$$

From (22) we have $(*): \deg(q_i \frac{\partial f_j}{\partial x_i}) \leq \deg(P) - 1$. We write again $q_i \frac{\partial f_j}{\partial x_i}$ in the form (21) and substitute it in (23). By degree conditions this process stops and at the end we get the equation (20).

Now let us prove that x^I generate the $\mathbb{C}[x_0]$-module freely. For every $x_0 = a$ fixed, let V_a be the specialization of V at $x_0 = a$. All V_a's are vector spaces of the same dimension and according to the above argument x^I generates all V_a's. For V_0 it is even a $\mathbb{C}$-basis and so x^I is a basis of all V_a's. If the elements of x^I are not $\mathbb{C}[x_0]$-independent then we have $C. \sum_{\beta \in I} C_\beta x^\beta = 0$ in V for some $C, C_\beta \in \mathbb{C}[x_0]$ and C_β's do not have common zeros. We take an a which is not a zero of C. We have $\sum_{\beta \in I} C_\beta(a) x^\beta = 0$ in V_a which is a contradiction. $\square$

Proposition 1. *For every strongly tame polynomial $f \in \mathbb{C}[x]$ the forms $\omega_\beta := x^\beta dx, \beta \in I$ (resp. $\eta_\beta := x^\beta \eta, \ \beta \in I$) form a $\mathbb{C}[t]$- basis of the Brieskorn module H'' (resp. H') of f.*

14

Proof. We first prove the statement for H''. The statement for $f = g$ is well-known (see for instance [AGV88]). Recall the definition of the degree of a form in §1. We write an element $\omega \in \Omega^{n+1}, \deg(\omega) = m$ in the form

$$\omega = \sum_{\beta \in I} p_\beta(g)\omega_\beta + dg \wedge d\psi, \ p_\beta \in \mathbb{C}[t], \ \psi \in \Omega^{n-1}, \ \deg(p_\beta(g)\omega_\beta) \leq m, \ \deg(d\psi) \leq m - d$$

This is possible because g is homogeneous. Now, we write the above equality in the form

$$\omega = \sum_{\beta \in I} p_\beta(f)\omega_\beta + df \wedge d\psi + \omega', \ \text{with } \omega' = \sum_{\beta \in I}(p_\beta(g) - p_\beta(f))\omega_\beta + d(g - f) \wedge d\psi$$

The degree of ω' is strictly less than m and so we repeat what we have done at the beginning and finally we write ω as a $\mathbb{C}[t]$-linear combination of ω_β's. The forms $\omega_\beta, \beta \in I$ are linearly independent because $\#I = \mu$ and μ is the dimension of $H^n(L_c, \mathbb{C})$ for a regular $c \in \mathbb{C} - C$. The proof for H' is similar and uses the fact that for $\eta \in \Omega^n$ one can write

$$(24) \qquad \qquad \qquad \eta = \sum_{\beta \in I} p_\beta(g)\eta_\beta + dg \wedge \psi_1 + d\psi_2$$

and each piece in the right hand side of the above equality has degree less than $\deg(\eta)$. □

The above proposition gives us an algorithm to write every element of H' (resp. H'') as a $\mathbb{C}[t]$-linear sum of η_β's (resp. ω_β's). We must find such an algorithm first for the case $f = g$, which is not hard to do (see [Mo06-2]). Note that if $\eta \in \Omega^n$ is written in the form (24) then

$$d\eta = \sum_{\beta \in I}(p_\beta(g)A_\beta + p'_\beta(g)g)\omega_\beta - dg \wedge d\psi_1$$

We specialize the module V at $x_0 = 1$ and use Lemma 4 and obtain the following fact: x^I form a basis for the Milnor vector space V_f of f. Let $F_t = F - t.x_0^d$.

Lemma 5. *Let $b \in \mathbb{C}\backslash C$. There is a map $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ such that the $\mathbb{C}$-vector space $\tilde{V} := \mathbb{C}[x_0, x]/ < \frac{\partial F_b}{\partial x_i} \mid i = 0, 1, \ldots, n + 1 >$ is freely generated by*

$$(25) \qquad \qquad \qquad \{x_0^{\beta_0}x^\beta, 0 \leq \beta_0 \leq d_\beta - 1, \beta \in I\}$$

In particular, the $\mathbb{C}(t)$-vector space

$$V' := \mathbb{C}(t)[x_0, x]/ < \frac{\partial F_t}{\partial x_i} \mid i = 0, 1, \ldots, n + 1 >$$

is freely generate by (25).

Proof. We consider the class Cl of all sets of the form (25) whose elements are linearly independent in $\tilde{V}$. For instance the one element set $\{x^{\beta'}\}$ is in this class. In this example $d_{\beta'} = 1$ and $d_\beta = 0, \forall \beta \in I - \{\beta'\}$. Since $\tilde{V}$ is a finite dimensional $\mathbb{C}$-vector space, Cl has only a finite number of elements and so we can take a maximal element A of Cl, i.e. there is no element of Cl containing A. We prove that A generates $\tilde{V}$ and so it is the desired set. Take a $\beta \in I$. We claim that $x^\beta x_0^k, \ k > d_\beta - 1$, can be written as a linear combination of the elements of A. The claim is proved by induction on k. For $k = d_\beta$, it is true because

15

A is maximal and $A \cup \{x^\beta x_0^{d_\beta}\}$ is of the form (25). Suppose that the claim is true for k. We write $x^\beta x_0^k$ as linear combination of elements of A and multiply it by x_0. Now we use the hypothesis of the induction for $k = d_\beta$ for the elements in the new summand which are not in A and get a linear combination of the elements of A. Since $\tilde{V} = V/ < \frac{\partial F_b}{\partial x_0} >$ and V is a $C[x_0]$-module generate by x^β's, we conclude that $x^\beta x_0^k, k \in \mathbb{N} \cup \{0\}$, $\beta \in I$ generate $\tilde{V}$ and so A generates $\tilde{V}$.

If there is a $\mathbb{C}(t)$-linear relation between the elements of (25) then we multiply it by a suitable element of $\mathbb{C}(t)$ and obtain a $\mathbb{C}[t]$-linear relation such that putting $t = b$ gives us a nontrivial relation in $\tilde{V}$. This proves the second part. $\square$

Remark 2. Lemma 5 implies that for all $c \in \mathbb{C}$, except a finite number of them which includes C and does not include b, the set (25) is a basis of the specialization of V' at $t = c$. The set $\tilde{C}$ of such exceptional values may be greater than C. To avoid such a problem we may try to prove the following fact which seems to be true:
(*) Let $\mathbb{C}[t]_C$ be the localization of $\mathbb{C}[t]$ on its multiplicative subgroup generated by $t-c, c \in C$. There is a function $\beta \in I \to d_\beta \in \mathbb{N} \cup \{0\}$ such that the $\mathbb{C}[t]_C$-module $V'' := \mathbb{C}[t]_C[x_0, x]/ < \frac{\partial F_t}{\partial x_i} \mid i = 0, 1, \ldots, n+1 >$ is freely generated by $\{x_0^{\beta_0} x^\beta, 0 \leq \beta_0 \leq d_\beta-1, \beta \in I\}$.
In [Mo06-2] I have used another algorithm (different from the one in the proof of Lemma 5). The advantage of this algorithm is that it also determines whether the obtained basis of V' is a $\mathbb{C}[t]_C$ basis of V'' or not. A similar algorithm shows that one can take $d_\beta = d-1$ for a generic b.

Let f be a quasi-homogeneous polynomial of degree d. In this case $F_t = f - t.x_0^d$ and V' is generated by $\{x^\beta x_0^{\beta_0}, \beta \in I, 0 \leq \beta_0 \leq d - 2\}$. In fact it generates the $\mathbb{C}[t]_C$-module V'' freely and so $\tilde{C} = C = \{0\}$. The dimension of V' is $(d-1)\mu$ and so the Milnor number of $f - t.x_0^d$ is $(d-1)\mu$. Since the Milnor number is topologically invariant, we conclude that for an arbitrary strongly tame function f the dimension of V' is $(d-1)\mu$ and so

$$\sum_{\beta \in I} d_\beta = (d-1)\mu$$

Moreover, we have $0 < A_{(\beta, \beta_0)} = A_\beta + \frac{\beta_0+1}{d} < n + 2$ for all $\beta \in I$ and $0 \leq \beta_0 \leq d_\beta - 1$ and so

$$d_\beta < d(n + 2 - A_\beta)$$

Proof of Theorem 1: Since the dimensions of the pieces of the mixed Hodge structure of a smooth fiber L_c does not depend on the analytic structure, the equality $Gr_m^W H' = 0$, $m \neq n, n+1$ follows form Steenbrink's theorem for the quasi-homogeneous polynomials.

We use Lemma 5 and we obtain a basis $\{x_0^{\beta_0} x^\beta, 0 \leq \beta_0 \leq d_\beta - 1, \beta \in I\}$ of the $\mathbb{C}[t]_{\tilde{C}}$-module V''. Recall the notations introduced in §2 and $\tilde{C}$ in Remark 2. Theorem 2 and Lemma 3 imply that the residue of the forms

(26) $$\frac{x^\beta dx}{(f - c)^k}, \ A_\beta = k$$

in L_c form a basis of $Gr_F^{n+1-k} Gr_{n+1}^W H^n(L_c, \mathbb{C})$ (Residue map is morphism of weight -2 of mixed Hodge structures). By Theorem 2 for $\mathbb{P}^{(1,\alpha)}$ and the hypersurface $X_c : F - cx_0^d = 0$, $c \in \mathbb{C} \backslash \tilde{C}$ and Lemma 5

$$\frac{x_0^{\beta_0} x^\beta \eta_{(1,\alpha)}}{(F - cx_0^d)^k}, \ A_\beta + \frac{\beta_0 + 1}{d} = k, \ 0 \leq \beta_0 \leq d_\beta - 1$$

16

form a basis for $Gr_F^{n+2-k}Gr_{n+2}^W H^{n+1}(\mathbb{P}^{(1,\alpha)} - X_c, \mathbb{C})$. In the affine coordinate $\mathbb{C}^{n+1} \subset \mathbb{P}^{(1,\alpha)}$, these forms are

$$(27) \qquad \frac{x^\beta dx}{(f-c)^k}, \ A_\beta + \frac{1}{d} \le k \le A_\beta + \frac{d_\beta}{d}, \ k \in \mathbb{N}.$$

So the residues of the above forms at L_c form a basis of $Gr_F^{n+1-k}Gr_n^W H^n(L_c, \mathbb{C})$. We apply Lemma 2 to the meromorphic forms (26) and (27) and obtain the fact that the forms (2) (resp. the forms (3)) restricted to the fiber L_c, $c \in \mathbb{C}\backslash\tilde{C}$ form a basis of $Gr_F^{n+1-k}Gr_{n+1}^W H^n(L_c, \mathbb{C})$ (resp. $Gr_F^{n+1-k}Gr_n^W H^n(L_c, \mathbb{C})$). Note that $x^\beta dx = d(\frac{\eta_\beta}{A_\beta})$. $\qquad \square$

7 Examples and applications

In this section we give some examples of the polynomial f and discuss the result of the paper on them. The examples which we discuss are of the form $f(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} f_i(x_i)$, where f_i is a polynomial of degree m_i, $m_i \ge 2$ in one variable x_i and with leading coefficient one. Let d be the least common multiple of $m_i's$. We consider f in the weighted ring $\mathbb{C}[x], deg(x_i) = \frac{d}{m_i}$, $i = 1, 2 \ldots, n+1$. Then $deg(f) = d$ and the last homogeneous part of f is $g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$. The vector space $V = \mathbb{C}[x]/\text{Jacob}(f)$ has the following basis of monomials

$$x^\beta, \ \beta \in I := \{\beta \in \mathbb{Z}^{n+1} \mid 0 \le \beta_i \le m_i - 2\}$$

and $\mu = \#I = \Pi_{i=1}^{n+1}(m_i - 1)$. To calculate the dimensions of the pieces of $W_\bullet H'$ and $F^\bullet H'$, it is enough to do it for g. Because in the weighted projective compactification of $\mathbb{C}^{n+1}$ the fibers of f and g are obtained by smooth deformations of each others and the dimension of the pieces of a mixed Hodge structure is constant under smooth deformations (see [KK98] Chapter 2 §3). We obtain

$$dim(Gr_F^{n+1-k}Gr_{n+1}^W H') = \#\{\beta \in I \mid A_\beta = k\}$$

$$dim(Gr_F^{n+1-k}Gr_n^W H') = \#\{\beta \in I \mid k - 1 < A_\beta < k\}$$

where $A_\beta = \sum_{i=1}^{n+1} \frac{(\beta_i+1)}{m_i}$. Let P_i be the collection of zeros of $\frac{\partial f_i}{\partial x_i} = 0$, with repetitions according to the multiplicity, and $C_i = f_i(A_i)$. The set of singularities of f is $P = P_1 \times P_2 \times \cdots P_{n+1}$ and $\sum_{i=1}^{n+1} C_i = \{\sum_{i=1}^{n+1} c_i \mid c_i \in C_i\}$ is the set of critical values of f. The Milnor number of a singularity is the number of its repetition in P.

Before analyzing some examples, let us state a consequence of the Hodge conjecture in the context of this article. We assume that f is a strongly tame polynomial in $\mathbb{Q}[x]$ and t is an algebraic number. The Brieskorn module can be redefined over $\mathbb{Q}$ and it turns out that the Gauss-Manin connection is also defined over $\mathbb{Q}$. If the Hodge conjecture is true then a Hodge cycle $\delta \in H_n(L_t, \mathbb{Q})$ satisfies the following property: For any polynomial differential n-form $\omega \in W_n H'$ (defined over $\mathbb{Q}$) we have

$$(28) \qquad \int_\delta \omega \in (2\pi i)^{\frac{n}{2}} \bar{\mathbb{Q}}$$

(See Proposition 1.5 of Deligne's lecture [DMOS]). Such a property is proved for Abelian varieties of CM-type by Deligne. Since the main difficulty of the Hodge conjecture lies on construction of algebraic cycles, the above statement seems to be much easier to treat than the Hodge conjecture itself.

Example 1. ($f = g = x_1^{m_1} + x_2^{m_2} + \cdots + x_{n+1}^{m_{n+1}}$,). Let $G := \Pi_{i=1}^{n+1} G_{m_i}$, where $G_{m_i} := \{\epsilon_{m_i}^k \mid k = 0, 1, \ldots, m_i - 1\}$ and $\epsilon_{m_i} := e^{\frac{2\pi\sqrt{-1}}{m_i}}$ is a primitive root of the unity. The group G acts on each fiber L_c in the following way:

$$g : L_c \to L_c, \ (x_1, x_2, \ldots, x_{n+1}) \to (g_1 x_1, g_2 x_2, \ldots, g_{n+1} x_{n+1})$$

where $g = (g_1, g_2, \ldots, g_{n+1})$ is used for both a vector and a map. Let $I' = \Pi_{i=1}^{n+1}(G_{m_i} - \{1\})$. We have the one to one map

$$(29) \qquad\qquad I \to I', \ \alpha \to (\epsilon_{m_1}^{\alpha_1+1}, \epsilon_{m_2}^{\alpha_2+1}, \ldots, \epsilon_{m_{n+1}}^{\alpha_{n+1}+1})$$

and so we identify I' with I using this map. Fix a cycle $\delta \in H_n(L_c, \mathbb{Q})$. We have

$$g^* \omega_\beta = g^{\beta+1} \omega_\beta = \epsilon^{(\alpha+1)(\beta+1)} \omega_\beta$$

where

$$\epsilon^{(\alpha+1)(\beta+1)} := \epsilon_{m_1}^{(\alpha_1+1)(\beta_1+1)} \epsilon_{m_2}^{(\alpha_2+1)(\beta_2+1)} \cdots \epsilon_{m_{n+1}}^{(\alpha_{n+1}+1)(\beta_{n+1}+1)}$$

and g corresponds to α by (29). We have

$$(30) \qquad\qquad \int_{g_*\delta} \omega_\beta = \int_\delta g^* \omega_\beta = \epsilon^{(\alpha+1)(\beta+1)} \int_\delta \omega_\beta$$

Since $\cup_{\beta \in I}\{\delta \in H_n(L_c, \mathbb{Q}) \mid \int_\delta \omega_\beta = 0\}$ does not cover $H_n(L_c, \mathbb{Q})$, we take a cycle $\delta \in H_n(L_c, \mathbb{Q})$ such that $\int_\delta \omega_\beta \neq 0$, $\forall \beta \in I$. Therefore the period matrix P in this example is of the form $E.T$, where $E = [\epsilon^{(\alpha+1)(\beta+1)}]$ and T is the diagonal matrix with $\int_\delta \omega_\beta$ in the $\beta \times \beta$ entry. Now the $\omega_\beta, \beta \in I$ form a basis of H' and so the period matrix has non zero determinant. In particular the space of Hodge cycles in $H_n(L_c, \mathbb{Z})$ corresponds to the solutions of

$$(31) \qquad\qquad B.[\epsilon^{(\alpha+1)(\beta+1)}]_{\alpha \in I, \beta \in I_h} = 0$$

where B is a $1 \times \mu$ matrix with integer entries and $I_h = \{\beta \in I \mid A_\beta \notin \mathbb{N}, \ A_\beta < \frac{n}{2}\}$. This gives an alternative approach for the description of Hodge cycles for the Fermat variety given by Katz, Ogus and Shioda (see [Sh79]). Note that in their approach one gives an explicit basis of the $\mathbb{C}$-vector space generated by Hodge cycles and the elements of such a basis are not Hodge cycles. In the description (31) one can find easily a basis of the $\mathbb{Q}$-vector space of Hodge cycles. Even if the Hodge conjecture is proved (or disproved), the question of constructing an algebraic cycle just with its topological information B obtained from (31) will be another difficult problem in computational algebraic geometry.

For computations with the next example, we have used SINGULAR [GPS03].

Example 2. ($f = x_1^3 + x_2^3 + \cdots + x_5^3 - x_1 - x_2$) In this example $g = x_1^3 + x_2^3 + \cdots + x_5^3$, $I = \{0, 1\}^5$ and $S(t) = 27t^3 - 16t$. The statement (*) in Remark 2 is true for

$$d_\beta = \begin{cases} 4 & \beta_1 = \beta_2 = 0 \\ 2 & \beta_1 = 0, \ \beta_2 = 1 \\ 1 & \text{otherwise} \end{cases}.$$

and so the above data works for all regular values of f. This follows from the facts that a standard basis of the ideal $\mathrm{Jacob}(F_t)$, where F_t is the homogenization of $f - t$, is given by

$$2x_1 x_0 + 2x_2 x_0 + 3t x_0^2, x_5^2, x_4^2, x_3^2, 3x_2^2 - x_0^2, 3x_1^2 - x_0^2, 4x_2 x_0^2 + 3t x_0^3, x_0^4$$

18

and we have

$$S(t)x_0^4 = (\frac{-16}{3}x_2x_0 + 4tx_0^2)(3x_1^2 - x_0^2) + (\frac{16}{3}x_2x_0 + 12tx_0^2)(3x_2^2 - x_0^2)+$$

$$(-8x_1x_2 + 8x_2^2 + 6tx_1x_0 + 6tx_2x_0 - 9t^2x_0^2)(-2x_1x_0 - 2x_2x_0 + (-3t)x_0^2)$$

$$t(4x_2x_0^2 + 3tx_0^3) = \frac{4}{9}x_0(3x_1^2 - x_0^2) - \frac{4}{9}x_0(3x_2^2 - x_0^2) + (\frac{2}{3}x_1 - \frac{2}{3}x_2 - tx_0)(-2x_1x_0 - 2x_2x_0 - 3tx_0^2).$$

The data $d_\beta = d - 1 = 2$ does not work for the values $b = \pm(2/3)^{\frac{3}{2}}$. In fact for arbitrary t we have $x_0x_1x_2 = (\frac{9}{8}t^2 - \frac{1}{3})x_0^3$ in V', where V' is defined in Lemma 5. Now $\nabla^2(\eta)$ is a basis of $Gr_F^3 Gr_4^W H_C'$. We have

$$\nabla^2(\eta) = \frac{10}{3S(t)^2}((972t^2 - 192)x_1x_2\eta + (-405t^3 - 48t)x_2\eta + (-405t^3 - 48t)x_1\eta + (243t^4 - 36t^2 + 64)\eta)$$

which implies the statement in the Introduction.

It is remarkable that the integrals $\int_\delta \nabla^2(\eta)$, $\delta \in H_4(L_t, \mathbb{Q})$ satisfy the Picard-Fuchs equation

(32) $$(27t^3 - 16t)y'' + (81t^2 - 16)y' + 15ty = 0$$

It is a pull-back of a Gauss hypergeometric equation and so the integral $\int_\delta \nabla^2(\eta)$ can be expressed in terms of Gauss hypergeometric series. Since the Hodge conjecture is known for cubic hypersurfaces of dimension 4 by C. Clemens, J. P. Murre and S. Zucker (see [Zu77]), one can get some algebraic relations between the values of such functions on algebraic numbers. The philosophy of using geometry and obtaining algebraic values of special functions goes back to P. Deligne, F. Beukers, J. Wolfart and many others. In [MR06] we have shown that up to a constant, the periods $\int_\delta \nabla^2\eta$, $\delta \in H_4(L_t, \mathbb{Q})$ reduce to the periods of the differential form $\frac{dx}{y}$ on the elliptic curve $E_t : y^2 = x^3 - 3x + z$, $z := 2 - \frac{27}{4}t^2$. It is an interesting observation that E_b associated to the fiber L_b, $b = \pm(2/3)^{\frac{3}{2}}$, for which the data $d_\beta = d - 1$ does not work, is CM. Also the fiber $L_{(2/3)^{\frac{3}{2}}}$ is mapped to $L_{-(2/3)^{\frac{3}{2}}}$ under the automorphism $(x_1, x_2, x_3, x_4, x_5) \mapsto (-x_1, -x_2, \epsilon x_3, \epsilon x_4, \epsilon x_5)$ of the family $f = t$. For the effect of the automorphisms on variation of Hodge structures the reader is referred to [Us81]. We have shown that the value of the schwarz function $D(0, 0, 1|z) := -e^{-\pi i \frac{5}{6}} \frac{F(\frac{5}{6}, \frac{1}{6}, 1|z)}{F(\frac{5}{6}, \frac{1}{6}, 1|1-z)}$ belongs to $\mathbb{Q}(\zeta_3)$ at some $z \in \bar{\mathbb{Q}}$ if and only if

$$F(\frac{5}{6}, \frac{1}{6}, 1|z) \sim \frac{1}{\pi^2}\Gamma(\frac{1}{3})^3, \ F(\frac{5}{6}, \frac{1}{6}, 1|1-z) \sim \frac{1}{\pi^2}\Gamma(\frac{1}{3})^3$$

where $a \sim b$ means that $\frac{a}{b} \in \bar{\mathbb{Q}}$.

References

[AGV88] Arnold, V. I. and Gusein-Zade, S. M. and Varchenko, A. N.: Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals. Monographs in Mathematics, 83. Birkhäuser Boston, Inc., Boston, MA, 1988.

[Br70] Brieskorn, E.: Die Monodromie der isolierten Singularitäten von Hyperflächen. Manuscripta Math. 2 1970 103–161.

[BH61] Borel, A. and Haefliger, A.: La classe d'homologie fondamentale d'un espace analytique. Bull. Soc. Math. France 89 1961 461–513.

[Ch96] Chéniot, D.: Vanishing cycles in a pencil of hyperplane sections of a non-singular quasi-projective variety. Proc. London Math. Soc. (3) 72 (1996), no. 3, 515–544.

[De71] Deligne, P.: Théorie de Hodge. II. Inst. Hautes Études Sci. Publ. Math. No. 40, (1971), 5–57.

[DMOS] Deligne, P. and Milne, J. S. and Ogus, A. and Shih, K.: Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics, 900, Springer-Verlag, 1982.

[DN01] Dimca, A. and Némethi, A.: On the monodromy of complex polynomials. Duke Math. J. 108 (2001), no. 2, 199–209.

[Do82] Dolgachev, I.: Weighted projective varieties. Group actions and vector fields (Vancouver, B.C., 1981), 34–71, Lecture Notes in Math., 956, Springer, Berlin, 1982.

[El86] El Zein, F.: Théorie de Hodge des cycles évanescents. Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 1, 107–184.

[Ga98] Gavrilov, L.: Petrov modules and zeros of Abelian integrals. Bull. Sci. Math. 122 (1998), no. 8, 571–584.

[Ga02] Gavrilov, L.: The infinitesimal 16th Hilbert problem in the quadratic case. *Invent. Math.*, 143(3):449–497, 2001.

[GMV94] Green, M. and Murre, J. and Voisin, C.: Algebraic cycles and Hodge theory. Lecture Notes in Mathematics, 1594. Springer-Verlag, Berlin, 1994.

[GPS03] Greuel,G. M. and Pfister, G. and Schönemann, H.: Singular 2.0.4. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2001). http://www.singular.uni-kl.de.

[Gr69] Griffiths, P. A.: On the periods of certain rational integrals. I, II. Ann. of Math. (2) 90 (1969), 460-495 ;ibid. (2) 90 1969 496–541.

[Gr66] Grothendieck, A.: On the de Rham cohomology of algebraic varieties. Inst. Hautes tudes Sci. Publ. Math. No. 29, 1966 95–103.

[KK98] Kulikov, V.S. and Kurchanov, P. F.: Complex algebraic varieties: periods of integrals and Hodge structures. Algebraic geometry, III, 1–217, 263–270, Encyclopaedia Math. Sci., 36, Springer, Berlin, 1998.

[Mo04-1] Movasati, H.: Abelian integrals in holomorphic foliations, Revista Matemática Iberoamericana, Vol. 20, issue 1 (2004).

[Mo04-2] Movasati, H.: Center conditions: rigidity of logarithmic differential equations. J. Differential Equations 197 (2004), no. 1, 197–217.

[Mo06-1] Movasati, H.: Relative cohomology with respect to a Lefschetz pencil, J. Reine Angew. Math., Issue 594, 2006.

[Mo06-2] Movasati, H.: Calculation of mixed Hodge structures, Gauss-Manin connections and Picard-Fuchs equations, To appear in the proceeding of Sao Carlos Conference at CIRM, Brikhauser. Together with the Library foliation.lib written in Singular and available at www.impa.br/~hossein.

[MR06] Movasati, H., Reiter, S.: Hypergeometric series and Hodge cycles of four dimensional cubic hypersurfaces, to appear in Int. Jou. Number Theory, Vol 2, No 3, 2006.

[Pi97] Pink R.: Hodge Structures over Function Fields, preprint 1997.

[Sa99] Sabbah C.: Hypergeometric period for a tame polynomial. C. R. Acad. Sci. Paris. I Math. 328 (1999), no. 7, 603–608.

[ST85] Scherk, J. and Steenbrink, J. H. M.: On the mixed Hodge structure on the cohomology of the Milnor fibre. Math. Ann. 271 (1985), no. 4, 641–665.

[Sc73] Schmid, W.: Variation of Hodge structure: the singularities of the period mapping. Invent. Math. 22 (1973), 211–319.

[Sc05] Schulze, M.: Good bases for tame polynomials, Jou. Sym. Comp. 39,1 (2005), 103-126, math.AG/0306120.

[Sh79] Shioda, T.: The Hodge conjecture for Fermat varieties. Math. Ann. 245 (1979), no. 2, 175–184.

[St77] Steenbrink, J.: Intersection form for quasi-homogeneous singularities. Compositio Math. 34 (1977), no. 2, 211–223.

[St76] Steenbrink, J.: Mixed Hodge structure on the vanishing cohomology. Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[SZ85] Steenbrink, J. and Zucker, S.: Variation of mixed Hodge structure. I. Invent. Math. 80 (1985), no. 3, 489–542.

[Us81] Usui, S.: Effect of automorphisms on variation of Hodge structures. J. Math. Kyoto Univ. 21 (1981), no. 4, 645–672.

[Va80] Varčenko, A. N.: Asymptotic Hodge structure on vanishing cohomology, Izv. Akad. Nauk SSSR Ser. Mat., 45, 1981, 3, 540-591, 688.

[Zu77] Zucker, S.: The Hodge conjecture for cubic fourfolds. *Compositio Math.*, 34(2):199–209, 1977.

Instituto de Matemática Pura e Aplicada, IMPA,
Estrada Dona Castorina, 110,
22460-320, Rio de Janeiro,
RJ, Brazil.
Email: `hossein@impa.br`